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Dual pairs techniques in H^*-theories

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Abstract

This work is a version for Jordan pairs, of a previous result for Jordan algebras given in Rodriguez (1988). However, the tools we use are completely different from those in Rodriguez (1988). A Jordan H^*-pair is (in a sense) a complicated algebraic object enriched with a Hilbert space structure which is well related to its algebraic structure. In this work we describe a certain class of Jordan H^*-pairs by forgetting their Hilbert space structure and starting with the remaining purely algebraic information available on it. More precisely, if $((R^+,R^-),\langle\,\rangle)$ is an associative pair such that $((R^+,R^-)^J,\{\,\})$ with $\{x,y,z\}:=\langle x,y,z\rangle+\langle z,y,x\rangle$ is a topologically simple Jordan H^*-pair, then R can be endowed of an (associative) H^*-pair structure such that its H^*-symmetrized agrees with the Jordan H^*-pair R^J.

Let $A=(A^+,A^-)$ be a pair of modules over a commutative unitary ring K, and $\langle,,\rangle : A^\sigma \times A^{-\sigma} \times A^\sigma \to A^\sigma$, two trilinear maps such that $(x,y,z) \mapsto \langle x,y,z\rangle$ for $\sigma \in \{+,-\}$. Then A is called an *associative pair* if the following identities are satisfied:

$$\langle\langle x,y,z\rangle,u,v\rangle = \langle x,\langle y,z,u\rangle,v\rangle = \langle x,y,\langle z,u,v\rangle\rangle$$

for $x,z,v \in A^\sigma$ and $y,u \in A^{-\sigma}$.

Let us see a first example of an associative pair.

A dual pair of vector spaces over a division K-algebra Δ is a couple (X,X') such that X is a left Δ-vector space, X' is a right Δ-vector space and there is a non-degenerate bilinear form $f: X \times X' \to \Delta$. One can consider the X'-topology of X (and the X-topology of X'), and then define $L(X,X')$ as the set of all continuous linear maps from X to X'. In the same way, if we have two dual pairs (X,X') and (Y,Y'),

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one can consider the set $L(X,Y)$ (and $F(X,Y)$ the subset of all finite rank elements of $L(X,Y)$).

Any subpair of $(L(X,Y), L(Y,X))$ containing $(F(X,Y), F(Y,X))$ with the triple products $\langle x,y,z\rangle^\sigma := xyz$, is a prime associative pair with nonzero socle (see [3]).

Let $A=(A^+,A^-)$ be a pair of K-modules and

$$Q^\sigma : A^\sigma \to \hom_K(A^{-\sigma},A^\sigma)$$

two quadratic operators for $\sigma\in\{+,-\}$. We define the trilinear operators $\{,,\}^\sigma : A^\sigma \times A^{-\sigma}\times A^\sigma \to A^\sigma$ and the bilinear operator $D^\sigma : A^\sigma \times A^{-\sigma} \to End(A^\sigma)$ as $\{x,y,z\}^\sigma = D^\sigma(x,y)z := Q^\sigma(x+z)y - Q^\sigma(x)y - Q^\sigma(z)y$, for $x,z\in A^\sigma$, $y\in A^{-\sigma}$ and $\sigma\in\{+,-\}$. We will say that $A=(A^+,A^-)$ is a *Jordan pair* if the next identities and its linearizations are true:

$$D^\sigma(x,y)Q^\sigma(x) = Q^\sigma(x)D^{-\sigma}(y,x),$$
$$D^\sigma(Q^\sigma(x)y,y) = D^\sigma(x,Q^{-\sigma}(y)x),$$
$$Q^\sigma(Q^\sigma(x)y) = Q^\sigma(x)Q^{-\sigma}(y)Q^\sigma(x)$$

for $x,z\in A^\sigma$, $y\in A^{-\sigma}$ and $\sigma\in\{+,-\}$.

If A is an associative pair, then A^J will denote the *symmetrized* Jordan pair of A, that is, the Jordan pair whose underlying K-module agrees with that of A, and whose quadratic operators are given by $Q^\sigma(x)y = \langle x,y,x\rangle^\sigma$. Let $A=(A^+,A^-)$, $B=(B^+,B^-)$ be K-pairs. A couple $f=(f^+,f^-)$, $f^\sigma : A^\sigma\to B^\sigma$ of K-linear mappings will be called an *homomorphism* of the given pairs when $f^\sigma(\langle x,y,z\rangle) = \langle f^\sigma(x), f^{-\sigma}(y), f^\sigma(z)\rangle$ with $x,z\in A^\sigma$ and $y\in A^{-\sigma}$. The definitions of *epimorphism*, *monomorphism* and *isomorphism* are the usual ones. The opposite pair A^{op} of the pair $A=(A^+,A^-)$ is the pair (A^-,A^+) with the same triple products. An *anti-automorphism* from A to B is a K-linear mapping $v=(v^+,v^-)$ from the pair A to the pair B^{op} satisfying $v^\sigma(\langle x,y,z\rangle) = \langle v^\sigma(z), v^{-\sigma}(y), v^\sigma(x)\rangle$ for all $x,z\in A^\sigma$ and $y\in A^{-\sigma}$. An anti-automorphism $v=(v^+,v^-)$ of the pair will be called *involutive* if $v^\sigma v^{-\sigma} = Id$. Let $A=(A^+,A^-)$ be a pair over K. If $B=(B^+,B^-)$ is a couple of submodules $B^\sigma\subset A^\sigma$, then B will be called a *subpair* of A if $\langle B^\sigma,B^{-\sigma},B^\sigma\rangle\subset B^\sigma$. A couple of K-submodules $I=(I^+,I^-)$, $I^\sigma\subset A^\sigma$ is said to be an *inner ideal* iff $\langle x^\sigma,A^{-\sigma},x^\sigma\rangle\subset I^\sigma$ for all $x^\sigma\in I^\sigma$. An *ideal* $I=(I^+,I^-)$ of A is a couple of K-submodules such that $\langle I^\sigma,A^{-\sigma},A^\sigma\rangle, \langle A^\sigma,I^{-\sigma},A^\sigma\rangle, \langle A^\sigma,A^{-\sigma},I^\sigma\rangle \subset I^\sigma$. A pair A will be called simple iff $\langle A^\sigma,A^{-\sigma},A^\sigma\rangle\neq 0$ and its only ideals are 0 and A. Let $A=(A^+,A^-)$ be a complex pair and $*=(*^+,*^-)$ a couple of conjugate-linear mappings $*^\sigma : A^\sigma\to A^{-\sigma}$ for which $*^\sigma\circ *^{-\sigma} = Id$ and $\langle x^\sigma,y^{-\sigma},z^\sigma\rangle^{*^\sigma} = \langle (z^\sigma)^{*^\sigma}, (y^{-\sigma})^{*^{-\sigma}}, (x^\sigma)^{*^\sigma}\rangle$ for x^σ, $z^\sigma\in A^\sigma$ and $y^{-\sigma}\in A^{-\sigma}$. Then $*=(*^+,*^-)$ is called an *involution* of A. We say that A is an *H^*-pair* if A^+ and A^- are also Hilbert spaces over the complex numbers with inner products $(\cdot\mid\cdot)_\sigma : A^\sigma\times A^\sigma\to\mathbb{C}$, endowed with an involution $*=(*^+,*^-)$ such that

$$(\langle x^\sigma,y^{-\sigma},z^\sigma\rangle \mid t^\sigma)_\sigma = (x^\sigma \mid \langle t^\sigma,(z^\sigma)^{*^\sigma},(y^{-\sigma})^{*^{-\sigma}}\rangle)_\sigma$$
$$= (y^{-\sigma}\mid\langle (x^\sigma)^{*^\sigma},t^\sigma,(z^\sigma)^{*^{-\sigma}}\rangle)_{-\sigma} = (z^\sigma\mid\langle (y^{-\sigma})^{*^{-\sigma}},(x^\sigma)^{*^\sigma},t^\sigma\rangle)_\sigma$$

for $x^\sigma, z^\sigma, t^\sigma \in A^\sigma$ and $y^{-\sigma} \in A^{-\sigma}$. For any associative H^*-pair A, its symmetrized Jordan pair A^J is a Jordan H^*-pair with the same involution and inner product as A. We recall also that an H^*-pair A is said to be *topologically simple* when $\langle A^\sigma, A^{-\sigma}, A^\sigma \rangle \neq 0$ and its only closed ideals are $\{0\}$ and A.

Proposition 1. *Let $J = (J^+, J^-)$ be a topologically simple Jordan H^*-pair, then:*
(a) *J is non-degenerate.*
(b) *J is prime.*
(c) *$Soc(J) \neq 0$.*

Proof. The annihilator $Ann(J)$ of an H^*-pair is defined as the pair $(Ann^+(J), Ann^-(J))$ such that $x \in Ann^\sigma(J)$ if and only if $\{x, A^{-\sigma}, A^\sigma\} = 0$. In [1, Proof of Lemma 7], the relation

$$Z(J) := \{x \in J : \{x, J, x\} = 0\} = Ann(J),$$

where J is a Jordan triple system, is proved. This relation applied to the polarized Jordan triple system associated to J gives us (a).

As a consequence of $\{J^\sigma, J^{-\sigma}, J^\sigma\} \neq 0$ in a topologically simple H^*-pair we have (b).

The subpair generated by any x^+ and any x^- is associative. If we take $x^- := (x^+)^*$ we have an associative H^*-pair with isometric involution. If we suppose that $\langle x, x^*, x \rangle = 0$ for all $x \in J^+$, then we obtain immediately that J agrees with its annihilator. Hence there is some x^+ such that the associative H^*-pair generated by x^+ and $(x^+)^*$ does not agree with its annihilator. Any associative H^*-pair not agreeing with its annihilator and with a continuous involution, has a nonzero projection (polarizing, we can apply, for instance, the classification of complex H^*-ternary algebras given in [2]). Thus, we have that J has a nonzero projection e. Then the local algebra $J^\sigma_{e^{-\sigma}}$ is a Jordan H^*-algebra with zero annihilator (because of the global-to-local inheritance of nondegeneracy given in [7, Theorem 4.1]). It is known that a Jordan H^*-algebra with nonzero annihilator has a nonzero socle. Then J has a nonzero socle by the local-to-global inheritance result [7, Theorem 4.2]. Thus (c) is proved. □

We say that an associative algebra A with involution $*$ is an *$*$-envelope* for a Jordan triple system T if $T \subset H(A, *)$ and T generates A.

An $*$-envelope A is *$*$-tight* if every nonzero $*$-ideal $I = I^*$ of A satisfies $I \cap T \neq 0$.

We have to remember that every complex H^*-pair $A = (A^+, A^-)$ turns out to be a real H^*-pair restricting the field of scalars to $\mathbb{R}$ and defining the inner products as $(x^\sigma | y^\sigma)^{\mathbb{R}}_\sigma := \mathrm{Re}(x^\sigma | y^\sigma)_\sigma$. This real H^*-pair is denoted by $A^{\mathbb{R}} = ((A^+)^{\mathbb{R}}, (A^-)^{\mathbb{R}})$.

We shall need the following result:

Theorem 1 (D'Amour [4, Theorem B]). *For $i = 1, 2$ let T_i be a prime Jordan triple system with $Z(T_i) \neq 0$ (Zel'manov polynomial) and Z_2-graded $*$-tight algebra envelope*

A_i (no nonzero graded $*$-ideal of A_i misses T_i), $*$ a graded involution. Then any isomorphism $f: T_1 \to T_2$ extends uniquely to a graded $*$-isomorphism $F: A_1 \to A_2$.

Let $S = (\{A_i\}_{i \in I}, \{e_{j,i}\}_{i \leq j})$ be a directed system of pairs, we define the *direct limit*, $\varinjlim S$, as $(A, \{e_i\}_{i \in I})$ where A is a pair, and $e_i: A_i \to A$ are monomorphisms satisfying $e_i = e_j e_{j,i}$ for $i, j \in I$, $i \leq j$; and $(A, \{e_i\}_{i \in I})$ is universal in the usual sense. If every $A_i = (A_i^+, A_i^-)$, $i \in I$ is an H^*-pair we define the concept of directed system of H^*-pairs in a similar way but $e_{j,i}: A_i \to A_j$ is required to be an isometric $*$-monomorphism and moreover, there must be real positive numbers h, k such that for all $i \in I$:

(1) $\|(x^\sigma)^{*^\sigma}\| \leq k\|x^\sigma\|$, $x^\sigma \in A_i^\sigma$.

(2) $\|\langle x^\sigma, y^{-\sigma}, z^\sigma \rangle\| \leq h\|x^\sigma\| \cdot \|y^{-\sigma}\| \cdot \|z^\sigma\|$ for every $x^\sigma, z^\sigma \in A_i^\sigma$ and $y^{-\sigma} \in A_i^{-\sigma}$.

We say that $(A, \{e_i\}_{i \in I})$ is an *H^*-direct limit* of the direct limit of H^*-pairs S whenever A is an H^*-pair, each e_i is an isometric $*$-monomorphism, and for any couple $(X, \{t_i\}_{i \in I})$ in which X is an H^*-pair and

$$t_i : A_i \to X$$

isometric $*$-monomorphisms verifying $t_i e_{i,j} = t_j$, then there is a unique isometric $*$-monomorphism $t: A \to X$ such that $te_i = t_i$. This will be denoted as $A = \varinjlim_{H^*} S$. The problem of the existence of H^*-direct limits can be solved as in the case of H^*-algebras. In fact, any directed system of H^*-pairs S has an H^*-direct limit (see [8, 3.3]).

Theorem 2 (Main theorem). *Let $R = (R^+, R^-)$ be an associative pair such that its symmetrized*

$$J = (R^+, R^-)^J$$

is a topologically simple Jordan H^*-pair. Then J is the Jordan H^*-pair associated to a topologically simple associative H^*-pair.

Proof. J is topologically simple, hence R is prime. As $Soc(J) \neq 0$ (Proposition 1), then $Soc(R) \neq 0$. Therefore, there are dual pairs (X, X') and (Y, Y') such that R is a subpair of $(L(X, Y), L(Y, X))$ containing $(F(X, Y), F(Y, X))$.

If we consider the associative algebra

$$A = \begin{pmatrix} F(X)^{\mathbb{R}} & F(X,Y)^{\mathbb{R}} \\ F(Y,X)^{\mathbb{R}} & F(Y)^{\mathbb{R}} \end{pmatrix}$$

with the product

$$\begin{pmatrix} \alpha_1 & f_1 \\ g_1 & \beta_1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 & f_2 \\ g_2 & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 . \alpha_2 + f_1 \cdot g_2 & \alpha_1 \cdot f_2 + f_1 \cdot \beta_2 \\ g_1 \cdot \alpha_2 + \beta_1 \cdot g_2 & g_1 \cdot f_2 + \beta_1 \cdot \beta_2 \end{pmatrix},$$

we have that $A \oplus A^{\mathrm{op}}$ is a Z_2-graded δ-tight algebra envelope of $P(J)^{\mathbb{R}}$ (the polarized Jordan triple system associated to J) with $\delta(x, y) := (y, x)$. Applying Theorem 1, $*$

extends to an automorphism $*' : A \oplus A^{\mathrm{op}} \to A \oplus A^{\mathrm{op}}$ hence by an easy argument the original map $* : R \to R$ has two possibilities:

(a) $\langle a,b,c\rangle^* = \langle a^*,b^*,c^*\rangle$ or (b) $\langle a,b,c\rangle^* = \langle c^*,b^*,a^*\rangle$. The possibility (a) gives us a contradiction with the isomorphism between $R_{11}(e)$ and $M_n(C)$ described in [5, Ch. 1], $e = (e^+,(e^+)^*)$ being a nonzero projection of R.

In [5, Theorem 3], it is proved that $\{J_2(e)\} = \{R_{11}(e)^J\}$, the family of the (2)-Peirce spaces of J with inclusion, is a direct system of Jordan pairs and $Soc(J) = \varinjlim(\{R_{11}(e)^J\})$, Loos' result [6] can be refined in our case so as to find a direct system of H^*-subpairs $\{R_{11}(e)^J\}$, where e ranges in a suitable family of nonzero projections. Moreover, it is possible to prove that $\{R_{11}(e)^J\}$ and $\{R_{11}(e)\}$ are, with the inclusion, direct systems of Jordan H^*-pairs and associative H^*-pairs, respectively, and we have

$$J = \overline{Soc(J)} = \lim_{\longrightarrow_{H^*}} (\{R_{11}(e)^J\}) = \lim_{\longrightarrow_{H^*}} (\{R_{11}(e)\})^J = (R')^J,$$

R' being a topologically simple associative H^*-pair. Furthermore, it can be proved that R is an associative H^*-pair whose symmetrization is J. □

In fact, it is possible to prove the following result which would complement the previous one:

Consider a dual pair (X,Y) and J a Jordan H^*-subpair of

$$(H(L(X,Y),\sharp), H(L(Y,X),\sharp))$$

containing to $(H(F(X,Y),\sharp), H(F(Y,X),\sharp))$. Then J is the Jordan H^*-pair coming from topologically simple Jordan H^*-algebra or the Jordan H^*-pair coming from the symmetrization of certain ternary H^*-algebra.

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